

# Approximating Fixed Points of Nonexpansive Mappings by a Faster Iteration Process

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## Abstract

In this paper, we consider a new iteration process which is faster than all of Picard, Mann, Ishikawa and Agarwal et al. processes. We also prove some strong and weak convergence theorems for the class of nonexpansive mappings in Banach spaces.

## 1 Introduction and Preliminaries

Throughout this paper,  $\mathbb{N}$  denotes the set of all positive integers. Let  $C$  be a nonempty convex subset of a normed space  $E$ , and  $T : C \rightarrow C$  be a mapping. Then we denote the set of all fixed points of  $T$  by  $F(T)$ .  $T$  is called  $L$ -Lipschitzian if there exists a constant  $L > 0$  such that  $\|Tx - Ty\| \leq L\|x - y\|$  for all  $x, y \in C$ . An  $L$ -Lipschitzian is called contraction if  $L \in (0, 1)$ , and nonexpansive if  $L = 1$ .

We know that the Picard [1], Mann [2] and Ishikawa [3] iteration processes are defined respectively as:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = Tx_n, \quad n \in \mathbb{N}, \end{cases} \quad (1.1)$$

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad n \in \mathbb{N}, \end{cases} \quad (1.2)$$

and

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n, \\ y_n = (1 - \beta_n)x_n + \beta_nTx_n, \quad n \in \mathbb{N}, \end{cases} \quad (1.3)$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are in  $(0, 1)$ .

Recently, Agarwal, O'Regan and Sahu [4] have introduced the S-iteration process as follows:

$$\begin{cases} a_1 = a \in C, \\ a_{n+1} = (1 - \alpha_n)Ta_n + \alpha_n Tb_n, \\ b_n = (1 - \beta_n)a_n + \beta_n Ta_n, \quad n \in \mathbb{N}, \end{cases} \quad (1.4)$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are sequences in  $(0, 1)$ .

In [5] motivated by S-iteration process, the first author has introduced the normal S-iteration process as follows:

$$\begin{cases} t_1 = t \in C, \\ t_{n+1} = T((1 - \alpha_n)t_n + \alpha_n Tt_n), \quad n \in \mathbb{N}, \end{cases} \quad (1.5)$$

where  $\{\alpha_n\}$  is in  $(0, 1)$ .

In order to compare two fixed point iteration processes  $\{u_n\}$  and  $\{v_n\}$  that converge to a certain fixed point  $p$  of a given operator  $T$ , Rhoades [6] considered that  $\{u_n\}$  is better than  $\{v_n\}$  if

$$\|u_n - p\| \leq \|v_n - p\| \quad \text{for all } n \in \mathbb{N}.$$

Berinde [7] introduced a different formulation from that of Rhoades as below:

**Definition 1** *Let  $\{a_n\}$  and  $\{b_n\}$  be two sequences of positive numbers that converge to  $a$ , respectively  $b$ . assume that there exists*

$$l = \lim_{n \rightarrow \infty} \frac{|a_n - a|}{|b_n - b|}.$$

- a) *If  $l = 0$ , then it can be said that  $\{a_n\}$  converges to  $a$  faster than  $\{b_n\}$  converges to  $b$ .*
- b) *If  $0 < l < \infty$ , then it can be said that  $\{a_n\}$  and  $\{b_n\}$  have the same rate of convergence.*

In the sequel, whenever we talk about the rate of convergence, we refer to the above definition.

Recently, Agarwal et al. [4] showed that, for contractions, S-iteration process converges at a same rate as Picard iteration and faster than Mann iteration. Sahu [5] proved that this process converges at a rate faster than both Picard and Mann iterations for contractions, by giving a numerical example in support of his claim. After, Khan [8] showed that (1.5) converges at a rate faster than all of Picard (1.1), Mann (1.2) and Ishikawa (1.3) iterative processes for contractions.

Our purpose in this paper is to present a new iteration process that, for contractions, converges faster than both the S-iteration process and the normal S-iteration process. We also prove a strong convergence theorem with the help of this process for the class of nonexpansive mappings in general Banach spaces

and apply it to get a result in uniformly convex Banach spaces. Our iteration process for one mapping is as follows:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = Ty_n, \\ y_n = (1 - \alpha_n)z_n + \alpha_n Tz_n, \\ z_n = (1 - \beta_n)x_n + \beta_n Tx_n, \quad n \in \mathbb{N} \end{cases} \quad (1.6)$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are in  $(0, 1)$ .

**Remark 2** *i) The process (1.6) is indepent of all Picard, Mann, Ishikawa and S-iteration processes since  $\{\alpha_n\}$  and  $\{\beta_n\}$  are in  $(0, 1)$ .*

*ii) Even if it is allowed to take  $\beta_n = 0$  and  $\alpha_n = \beta_n = 0$  in the process (1.6), our process reduces to normal S-iteration (1.5) and Picard iteration (1.1) processes.*

We recall the following. Let  $S = \{x \in E : \|x\| = 1\}$  and let  $E^*$  be the dual of  $E$ , that is, the space of all continuous linear functional  $f$  on  $E$ . The space  $E$  has:

(i) *Gâteaux differentiable norm* if

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t},$$

exists for each  $x$  and  $y$  in  $S$ ;

(ii) *Fréchet differentiable norm* (see e.g. [9, 10]) if for each  $x$  in  $S$ , the above limit exists and is attained uniformly for  $y$  in  $S$  and in this case, it is also well-known that

$$\langle h, J(x) \rangle + \frac{1}{2} \|x\|^2 \leq \frac{1}{2} \|x + h\|^2 \leq \langle h, J(x) \rangle + \frac{1}{2} \|x\|^2 + b(\|h\|) \quad (1.7)$$

for all  $x, h$  in  $E$ , where  $J$  is the Fréchet derivative of the functional  $\frac{1}{2} \|\cdot\|^2$  at  $x \in X$ ,  $\langle \cdot, \cdot \rangle$  is the pairing between  $E$  and  $E^*$ , and  $b$  is an increasing function defined on  $[0, \infty)$  such that  $\lim_{t \downarrow 0} \frac{b(t)}{t} = 0$ ;

(iii) *Opial condition* [11] if for any sequence  $\{x_n\}$  in  $E$ ,  $x_n \rightharpoonup x$  implies that  $\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$  for all  $y \in E$  with  $y \neq x$ . Examples of Banach spaces satisfying Opial condition are Hilbert spaces and all spaces  $l^p$  ( $1 < p < \infty$ ). On the other hand,  $L^p[0, 2\pi]$  with  $1 < p \neq 2$  fail to satisfy Opial condition.

A mapping  $T : C \rightarrow E$  is demiclosed at  $y \in E$  if for each sequence  $\{x_n\}$  in  $C$  and each  $x \in E$ ,  $x_n \rightharpoonup x$  and  $Tx_n \rightarrow y$  imply that  $x \in C$  and  $Tx = y$ .

We will use the following lemmas in order to prove the our main results.

**Lemma 3** [12] Suppose that  $E$  is a uniformly convex Banach space and  $0 < p \leq t_n \leq q < 1$  for all  $n \in \mathbb{N}$ . Let  $\{x_n\}$  and  $\{y_n\}$  be two sequences of  $E$  such that  $\limsup_{n \rightarrow \infty} \|x_n\| \leq r$ ,  $\limsup_{n \rightarrow \infty} \|y_n\| \leq r$  and  $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = r$  hold for some  $r \geq 0$ . Then  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ .

**Lemma 4** [13] Let  $E$  be a uniformly convex Banach space and let  $C$  be a nonempty closed convex subset of  $E$ . Let  $T$  be a nonexpansive mapping of  $C$  into itself. Then  $I - T$  is demiclosed with respect to zero.

## 2 Rate of Convergence

In this section, we show that our process (1.6) converges faster than processes (1.4) and (1.5).

**Theorem 5** Let  $C$  be a nonempty closed convex subset of normed space  $E$ , let  $T$  be a contraction of  $C$  into itself. Suppose that each of iterative processes (1.4), (1.5) and (1.6) converges to the same fixed point  $p$  of  $T$  where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are such that  $0 < \lambda \leq \alpha_n, \beta_n < 1$  for all  $n \in \mathbb{N}$  and for some  $\lambda$ . Then the iterative process given by (1.6) converges faster than (1.4) and (1.5).

**Proof.** For S-iterative process (1.4), we have

$$\begin{aligned}
\|a_{n+1} - p\| &= \|(1 - \alpha_n)Ta_n + \alpha_n Tb_n - p\| \\
&= \|(1 - \alpha_n)(Ta_n - p) + \alpha_n(Tb_n - p)\| \\
&\leq (1 - \alpha_n) \|Ta_n - p\| + \alpha_n \|Tb_n - p\| \\
&\leq L[(1 - \alpha_n) \|a_n - p\| + \alpha_n \|b_n - p\|] \\
&= L[(1 - \alpha_n) \|a_n - p\| + \alpha_n \|(1 - \beta_n)a_n + \beta_n Ta_n - p\|] \\
&= L[(1 - \alpha_n) \|a_n - p\| + \alpha_n \|(1 - \beta_n)(a_n - p) + \beta_n(Ta_n - p)\|] \\
&\leq L[(1 - \alpha_n) \|a_n - p\| + \alpha_n(1 - \beta_n) \|a_n - p\| + \alpha_n \beta_n \|Ta_n - p\|] \\
&\leq L[(1 - \alpha_n) + \alpha_n(1 - \beta_n) + L\alpha_n \beta_n] \|a_n - p\| \\
&= L(1 - \alpha_n \beta_n (1 - L)) \|a_n - p\| \\
&\leq L(1 - \lambda^2(1 - L)) \|a_n - p\| \\
&\vdots \\
&\leq [L(1 - \lambda^2(1 - L))]^n \|a_1 - p\|.
\end{aligned}$$

Let  $k_n = [L(1 - \lambda^2(1 - L))]^n \|a_1 - p\|$ .  
From (1.5), we obtain that

$$\begin{aligned}
\|t_{n+1} - p\| &= \|T((1 - \alpha_n)t_n + \alpha_n Tt_n) - p\| \\
&\leq L \|(1 - \alpha_n)(t_n - p) + \alpha_n(Tt_n - p)\| \\
&\leq L[(1 - \alpha_n)\|t_n - p\| + \alpha_n L\|t_n - p\|] \\
&= L(1 - (1 - L)\alpha_n)\|t_n - p\| \\
&\leq L(1 - (1 - L)\lambda)\|t_n - p\| \\
&\vdots \\
&\leq [L(1 - (1 - L)\lambda)]^n \|t_1 - p\|.
\end{aligned}$$

Let  $l_n = [L(1 - (1 - L)\lambda)]^n \|t_1 - p\|$ .

Our process (1.6) gives

$$\begin{aligned}
\|x_{n+1} - p\| &= \|Ty_n - p\| \\
&\leq L\|y_n - p\| \\
&= L\|(1 - \alpha_n)z_n + \alpha_n Tz_n - p\| \\
&= L\|(1 - \alpha_n)(z_n - p) + \alpha_n(Tz_n - p)\| \\
&\leq L[(1 - \alpha_n)\|z_n - p\| + \alpha_n\|Tz_n - p\|] \\
&\leq L[(1 - \alpha_n)\|z_n - p\| + \alpha_n L\|z_n - p\|] \\
&= L(1 - (1 - L)\alpha_n)\|z_n - p\| \\
&= L(1 - (1 - L)\alpha_n)\|(1 - \beta_n)x_n + \beta_n Tx_n - p\| \\
&= L(1 - (1 - L)\alpha_n)\|(1 - \beta_n)(x_n - p) + \beta_n(Tx_n - p)\| \\
&\leq L(1 - (1 - L)\alpha_n)[(1 - \beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\|] \\
&\leq L(1 - (1 - L)\alpha_n)[(1 - \beta_n)\|x_n - p\| + \beta_n L\|x_n - p\|] \\
&= L(1 - (1 - L)\alpha_n)(1 - (1 - L)\beta_n)\|x_n - p\| \\
&\leq L(1 - (1 - L)\lambda)^2\|x_n - p\| \\
&\vdots \\
&\leq [L(1 - (1 - L)\lambda)^2]^n \|x_1 - p\|.
\end{aligned}$$

Let  $m_n = [L(1 - (1 - L)\lambda)^2]^n \|x_1 - p\|$ . Then

$$\frac{m_n}{k_n} = \frac{[L(1 - (1 - L)\lambda)^2]^n \|x_1 - p\|}{[L(1 - (1 - L)\lambda^2)]^n \|a_1 - p\|} = \left[ \frac{(1 - (1 - L)\lambda)^2}{1 - (1 - L)\lambda^2} \right]^n \frac{\|x_1 - p\|}{\|a_1 - p\|} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Thus  $\{x_n\}$  converges faster than  $\{a_n\}$  to  $p$ . Similarly

$$\frac{m_n}{l_n} = \frac{[L(1 - (1 - L)\lambda)^2]^n \|x_1 - p\|}{[L(1 - (1 - L)\lambda)]^n \|t_1 - p\|} = [1 - (1 - L)\lambda]^{2n} \frac{\|x_1 - p\|}{\|t_1 - p\|} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence  $\{x_n\}$  converges faster than  $\{t_n\}$  to  $p$ . ■

### 3 Convergence Theorems

In this section, we give some convergence theorems using our iteration process (1.6).

**Lemma 6** *Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$  and let  $T$  be a nonexpansive self mapping of  $C$ . Let  $\{x_n\}$  be defined by the iteration process (1.6) where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are in  $(0, 1)$  for all  $n \in \mathbb{N}$ . Then*

(i)  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for all  $p \in F(T)$ .

(ii)  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ .

**Proof.** Let  $p \in F(T)$ . Then

$$\begin{aligned}
 \|z_n - p\| &= \|(1 - \beta_n)x_n + \beta_n Tx_n - p\| \\
 &= \|(1 - \beta_n)(x_n - p) + \beta_n(Tx_n - p)\| \\
 &\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\| \\
 &\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\
 &= \|x_n - p\|,
 \end{aligned} \tag{1.8}$$

and so

$$\begin{aligned}
 \|x_{n+1} - p\| &= \|Ty_n - p\| \\
 &\leq \|y_n - p\| \\
 &= \|(1 - \alpha_n)z_n + \alpha_n Tz_n - p\| \\
 &= \|(1 - \alpha_n)(z_n - p) + \alpha_n(Tz_n - p)\| \\
 &\leq (1 - \alpha_n)\|z_n - p\| + \alpha_n\|Tz_n - p\| \\
 &\leq (1 - \alpha_n)\|z_n - p\| + \alpha_n\|z_n - p\| \\
 &= \|z_n - p\| \\
 &\leq \|x_n - p\|.
 \end{aligned}$$

This shows that  $\{\|x_n - p\|\}$  is decreasing, and this proves part (i). Let

$$\lim_{n \rightarrow \infty} \|x_n - p\| = c. \tag{1.9}$$

Now,  $\|Tx_n - p\| \leq \|x_n - p\|$  implies that

$$\limsup_{n \rightarrow \infty} \|Tx_n - p\| \leq c. \tag{1.10}$$

Since  $\|x_{n+1} - p\| \leq \|z_n - p\|$ , therefore

$$\liminf_{n \rightarrow \infty} \|x_{n+1} - p\| \leq \liminf_{n \rightarrow \infty} \|z_n - p\|,$$

so

$$c \leq \liminf_{n \rightarrow \infty} \|z_n - p\| \quad (1.11)$$

On the other hand, (1) implies that

$$\limsup_{n \rightarrow \infty} \|z_n - p\| \leq c. \quad (1.12)$$

From (1.11) and (1.12)

$$\lim_{n \rightarrow \infty} \|z_n - p\| = c.$$

Hence, this implies that

$$c = \lim_{n \rightarrow \infty} \|z_n - p\| = \lim_{n \rightarrow \infty} \|(1 - \beta_n)(x_n - p) + \beta_n(Tx_n - p)\|. \quad (1.13)$$

Using (1.9), (1.10), (1.13) and Lemma 3, we obtain

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0.$$

■

**Lemma 7** *Assume that all the conditions of Lemma 6 are satisfied. Then, for any  $p_1, p_2 \in F(T)$ ,  $\lim_{n \rightarrow \infty} \langle x_n, J(p_1 - p_2) \rangle$  exists; in particular,  $\langle p - q, J(p_1 - p_2) \rangle = 0$  for all  $p, q \in \omega_w(x_n)$ , the set of all weak limits of  $\{x_n\}$ .*

**Proof.** The proof of this lemma is the same as the proof of Lemma 2.3 of [14]. So, we omit it here. ■

We now give our weak convergence theorem.

**Theorem 8** *Let  $E$  be a uniformly convex Banach space and let  $C, T$  and  $\{x_n\}$  be taken as in Lemma 6. Assume that (a)  $E$  satisfies Opial's condition or (b)  $E$  has a Fréchet differentiable norm. If  $F(T) \neq \emptyset$ , then  $\{x_n\}$  converges weakly to a fixed point of  $T$ .*

**Proof.** From (i) in Lemma 6, we know that  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for all  $p \in F(T)$ . Thus  $\{x_n\}$  is bounded. Since  $E$  is uniformly convex,  $\{x_n\}$  has a subsequence  $\{x_{n_k}\}$  which converges weakly in  $C$ . We prove that  $\{x_n\}$  has a unique weak subsequential limit in  $F(T)$ . For this, let  $u$  and  $v$  be weak limits of subsequences  $\{x_{n_i}\}$  and  $\{x_{n_j}\}$  of  $\{x_n\}$ , respectively. By Lemma 6,  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$  and  $I - T$  is demiclosed with respect to zero by Lemma 4; therefore, we obtain  $Tu = u$ . Again, in the same manner, we can prove that  $v \in F(T)$ . Next, we prove the uniqueness. To this end, first assume (a) is true.

If  $u$  and  $v$  are distinct, then by Opial's condition,

$$\begin{aligned}
\lim_{n \rightarrow \infty} \|x_n - u\| &= \lim_{n_i \rightarrow \infty} \|x_{n_i} - u\| \\
&< \lim_{n_i \rightarrow \infty} \|x_{n_i} - v\| \\
&= \lim_{n \rightarrow \infty} \|x_n - v\| \\
&= \lim_{n_j \rightarrow \infty} \|x_{n_j} - v\| \\
&< \lim_{n_j \rightarrow \infty} \|x_{n_j} - u\| \\
&= \lim_{n \rightarrow \infty} \|x_n - u\|.
\end{aligned}$$

This is a contradiction, so  $u = v$ . Next assume (b). By Lemma 7,  $\langle p - q, J(p_1 - p_2) \rangle = 0$  for all  $p, q \in \omega_w(x_n)$ . Therefore  $\|u - v\|^2 = \langle u - v, J(u_1 - v_2) \rangle = 0$  implies  $u = v$ . Consequently,  $\{x_n\}$  converges weakly to a point of  $F$  and this completes the proof. ■

A mapping  $T : C \longrightarrow C$ , where  $C$  is a subset of normed space  $E$ , is said to satisfy Condition (A) [15] if there exists a nondecreasing function  $f : [0, \infty) \longrightarrow [0, \infty)$  with  $f(0) = 0$ ,  $f(r) > 0$  for all  $r \in (0, \infty)$  such that  $\|x - Tx\| \geq f(d(x, F(T)))$  for all  $x \in C$ , where  $d(x, F(T)) = \inf \{\|x - p\| : p \in F(T)\}$ .

**Theorem 9** *Let  $E$  be a uniformly convex Banach space and let  $C, T$  and  $\{x_n\}$  be taken as in Lemma 6. Then  $\{x_n\}$  converges to a point of  $F(T)$  if and only if  $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$ .*

**Proof.** Necessity is obvious. Suppose that  $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$ . From (i) in Lemma 6, we know that  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists for all  $p \in F(T)$ , therefore  $\lim_{n \rightarrow \infty} d(x_n, F(T))$  exists. But by hypothesis,  $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$ , therefore we have  $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$ . We will show that  $\{x_n\}$  is a Cauchy sequence in  $C$ . Since  $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$ , for given  $\varepsilon > 0$ , there exists  $n_0$  in  $\mathbb{N}$  such that for all  $n \geq n_0$ ,

$$d(x_n, F(T)) < \frac{\varepsilon}{2}.$$

Particularly,  $\inf \{\|x_{n_0} - p\| : p \in F(T)\} < \frac{\varepsilon}{2}$ . Hence, there exists  $p^* \in F(T)$  such that  $\|x_{n_0} - p^*\| < \frac{\varepsilon}{2}$ . Now, for  $m, n \geq n_0$ ,

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \leq 2\|x_{n_0} - p^*\| < \varepsilon.$$

Hence  $\{x_n\}$  is a Cauchy sequence in  $C$ . Since  $C$  is closed in the Banach space  $E$ , there exists a point  $q$  in  $C$  such that  $\lim_{n \rightarrow \infty} x_n = q$ . Now  $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$  gives that  $d(q, F(T)) = 0$ . Since  $F$  is closed,  $q \in F(T)$ . ■

Note that this condition is weaker than the requirement that  $T$  is demicompact or  $C$  is compact, see [15]. Applying Theorem 9, we obtain a strong convergence of the process (1.6) under Condition (A) as follows.



**Theorem 10** *Let  $E$  be a uniformly convex Banach space and let  $C, T$  and  $\{x_n\}$  be as in Lemma 6. If  $T$  satisfies Condition (A), then  $\{x_n\}$  converges strongly to a fixed point of  $T$ .*

**Proof.** From Lemma 6, we know that

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0. \quad (1.14)$$

Using Condition (A) and (1.14), we get

$$\lim_{n \rightarrow \infty} f(d(x_n, F(T))) \leq \lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0,$$

That is,  $\lim_{n \rightarrow \infty} f(d(x_n, F(T))) = 0$ . Since  $f : [0, \infty) \rightarrow [0, \infty)$  is a nondecreasing function satisfying  $f(0) = 0$ ,  $f(r) > 0$  for all  $r \in (0, \infty)$ , therefore, we have

$$\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0.$$

Now all the conditions of Theorem 9 are satisfied, therefore, by its conclusion,  $\{x_n\}$  converges strongly to a fixed point of  $T$ . ■

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